

Partial-conjugates and Dimensionality of Posets

Shaofang Qi*

February 24, 2014

Abstract

The Pareto dominance relation of a preference profile is (the asymmetric part of) a partial order. For any integer n , the problem of the existence of an n -agent preference profile that generates the given Pareto dominance relation is to investigate the *dimension* of the partial order. We provide a characterization of a partial order having dimension n in general.

1 Introduction

Consider the Pareto dominance relation at a profile of strong preferences defined on a finite set of objects. If the Pareto relation is observed but we are ignorant about the preference profile, for an integer n , when the Pareto relation can be generated by an n -agent preference profile?

The following observation allows us to rephrase the question. The existence of an n -agent preference profile implies that the same Pareto dominance relation can also be generated by an $(n + 1)$ -agent preference profile: assign the additional agent to have the same preference relation as any one of the existing n agents.¹ We therefore ask that for any n , when the minimum number of individuals whose preference profile can generate a given Pareto dominance relation is (at most) n .

Sprumont (2001) and Echenique and Ivanov (2011) answer the question for $n = 2$, from different perspectives. Sprumont imposes a set of “regularity” conditions and works on a rich continuum of alternatives, which allows him to utilize a set of simple, and intuitive basic conditions as (part of) a characterization. Recently, Qi (2013) has extended Sprumont’s basic conditions to a characterization for the finite case. Echenique and Ivanov (2011) require no specific additional structures on preferences and focus on the case of a finite set of options; they convert the question into a graph-coloring problem. To address the analogous question for $n \geq 3$ is what motivates this work.

*E-mail: sqi@syr.edu. I thank Susan Gensemer and Jerry Kelly for helpful comments and discussions.

¹Demuyne (2013) also mentions this observation.

More generally, the question is equivalent to investigate the *dimension* of a *partial order*.² A *partial order* is a reflexive, antisymmetric, and transitive binary relation defined on a set of options.³ So a Pareto dominance relation plus the diagonal of the binary relation (i.e. with reflexivity) is a partial order. Dushnik and Miller (1941) introduce the concept of the dimension of a partial order, which is the minimum number of linear orders whose intersection is the partial order. The characterizations of 2-dimensional partial orders have been well-documented. Besides the work mentioned above, there have been other different characterizations for the 2-dimensional case (see for instance, Dushnik and Miller (1941), Baker et al. (1972), Kelly (1977), and Trotter and Moore (1976)). The problem of determining the dimension of a poset having dimension (at most) n for any fixed $n \geq 3$ is NP-complete (Yannakakis (1982)).

The characterization we build on is from Dushnik and Miller (1941) for 2-dimensional partial orders. They introduce the concept of conjugate of a partial order which is another partial order defined on the same set of options such that every two distinct options can be comparable by exactly one of the two partial orders. We extend their concept of conjugates, in two steps. We first introduce the concept of partial-conjugates which preserves the properties similar to those hold by conjugates except that the union of two partial-conjugates partial orders cannot compare all distinct options. To incorporate this “completeness” property, we then introduce a finite sequence of partial orders which have the partial-conjugates relation and the union of the partial orders of the sequence has every two distinct options comparable. Our main result provides a characterization, based on our extensions of conjugates, which generalizes Dushnik and Miller’s theorem about conjugates and dimension 2. Our characterization result is of an “existential” nature in the sense that we are not providing an algorithm that can help to determine the dimension of a poset.

The rest of the paper is organized as follows. Section 2 discusses notation and definitions. Since we extend Dushnik and Miller’s result, we present their related concept and theorem in Section 3. In Section 4, we introduce our concept of partial-conjugates along with other concepts, and present our characterization result. Section 5 concludes with a discussion.

2 Notation and Definitions

Let X be a nonempty, finite set. We call X the *ground set*, and use $|X|$ to denote the number of elements in X . Let Δ_X denote the diagonal of $X \times X$, that is, $\Delta_X := \{(x, x) : x \in X\}$. A *binary relation* R on X is a nonempty subset of $X \times X$, and we write xRy instead of $(x, y) \in R$. A binary relation R on X

²All terminologies will be formally defined in Section 2.

³Some authors require irreflexivity in defining partial orders (e.g. Dushnik and Miller (1941)). But since almost all later work on dimension theory imposes reflexivity, we follow them and define a partial order to be reflexive. For an exposition on dimension theory, see the book by Trotter (1992).

is *reflexive* if xRx for any $x \in X$, *complete* if either xRy or yRx or both for any $x, y \in X$, *antisymmetric* if xRy and yRx imply that x, y are identical for any $x, y \in X$, and *transitive* if xRy and yRz imply xRz for any $x, y, z \in X$.⁴ If R is both reflexive and transitive, we call it a *quasi-order*. An antisymmetric quasi-order is a *partial order*. (That is, a partial order is a reflexive, transitive, and antisymmetric binary relation.) A complete quasi-order is a *weak order*. (That is, a weak order is a complete and transitive binary relation.) A complete partial order is a *linear order*. (That is, a linear order is a complete, transitive, and antisymmetric binary relation.) In addition, “ xRy and yRz ” is shortened to “ $xRyRz$,” with a similar convention applied to any finite conjunctions. Let T_R denote the *transitive closure* of R : xD_Ry if and only if there exist a positive integer K and elements x_1, \dots, x_K such that $xDx_1Rx_2 \cdots Rx_K = y$. An ordered pair (X, R) is called a *partially ordered set*, or simply, a *poset*, if R is a partial order on X . Throughout the rest of this paper, a generic partial order is denoted by P . And we use \mathbb{R}^n to denote n -dimensional Euclidean space.

Let (X, P) be a poset and consider any elements $x, y \in X$. We say that x and y are *comparable in P* , or simply, *comparable*, if either xPy or yPx or both. Accordingly, we say x and y are *incomparable in P* , or simply, *incomparable*, if x and y are not comparable in P . We write xNy in P if x and y are incomparable in P . The *dual* of a partial order P on X is denoted by P^d and is defined by xP^dy if and only if yPx . The *dual* of a poset (X, P) is denoted by (X, P^d) . Finally, the *dimension* of a poset (X, P) , denoted $\dim(X, P)$, is the smallest number of linear orders (defined on X) whose intersection is P . It is obvious that a poset and its dual have the same dimensionality.

3 Conjugate and Dimension 2

Dushnik and Miller (1941) introduce the concept of *conjugate*, which we illustrate next:

Example 1 (Conjugate). Suppose $X = \{x, y, z\}$. Consider two partial orders P and Q in Figure 1, both of which are defined on X .

x	y	z	x	y	z
x	P	P	x	Q	Q
y		P	y	Q	Q
z		P	z		Q
Partial Order P			Partial Order Q		

Figure 1: a Partial Order and a Conjugate.

⁴Note that completeness implies reflexivity. Some authors define completeness only for any two distinct options.

P and Q are related in the following sense: (i) if any two distinct options is comparable in P (resp., Q), then it is incomparable in Q (resp., P); and (ii) every two distinct options are comparable in either P or Q . For example, for distinct options x, y , xPy but xNy in Q . For distinct options x, y ; y, z ; and x, z : xPy , yQz and xQz . Additionally, $P \cup Q$ is a linear order on X : besides containing the diagonal $\Delta_{X \times X}$, $x(P \cup Q)y(P \cup Q)z$.

Dushnik and Miller (1941) use *conjugate* to generalize the relationship of P and Q in Example 1.

Definition (Conjugate, Dushnik and Miller (1941)). Let (X, P) and (X, Q) be two posets with the same ground set. P and Q are called *conjugate* partial orders if every two distinct options of X is ordered in exactly one of them.

By definition, for two posets (X, P) and (X, Q) , if P and Q are conjugate partial orders, then P and Q^d are also conjugate partial orders, where Q^d is the dual of Q . The following lemma generalizes the implication of two conjugate partial orders in Example 1.

Lemma (Lemma 3.51, Dushnik and Miller (1941)). Let (X, P) and (X, Q) be two posets with the same ground set X . If P and Q are conjugate partial orders, then $P \cup Q$ is a linear order defined on X .

We summarize the properties of partial orders P and Q defined on X that are conjugates:

Condition 1 P and Q cannot both order the same two distinct options of X .

Condition 2 $P \cup Q$ is a linear order.

Condition 3 $P \cup Q^d$ is a linear order.

Dushnik and Miller provide three characterizations of 2-dimensional partial orders, one of which connects the dimensionality of 2 to the existence of conjugate. Our work extends their characterization to n -dimensional partial orders; for comparison, we present their result here.

Theorem (Theorem 3.61 (1) and (3), Dushnik and Miller (1941)). Let (X, P) be a poset. Then $\dim(X, P) \leq 2$ if and only if P has a conjugate partial order.

4 Partial-conjugate and Dimensionality

We extend the conjugate concept and use the extended concept to characterize n -dimensional partial orders in general. Our characterization has an intuition that relates to the natural order defined on a subset of \mathbb{R}^n . We use a poset (X, P) with $X \subseteq \mathbb{R}^3$ to illustrate.

Example 2. Let $X = \{(4, 2, 2), (2, 1, 4), (1, 4, 1), (5, 3, 6), (3, 6, 5), (6, 5, 3)\} \subseteq \mathbb{R}^3$. For convenience, we denote these six elements in X by letters a, b, c, x, y , and z :

a	$(4, 2, 2)$
b	$(2, 1, 4)$
c	$(1, 4, 1)$
x	$(5, 3, 6)$
y	$(3, 6, 5)$
z	$(6, 5, 3)$

When we need to specify the i th coordinate of an element a letter denotes, we use the subscript i for $i \in \{1, 2, 3\}$. For instance, $a = (a_1, a_2, a_3)$ where $a_1 = 4$, $a_2 = 2$, and $a_3 = 2$. Consider an order P on X such that the diagonal $\Delta_X \subseteq P$ and for distinct options $u, v \in X$, uPv if and only if $u_i > v_i$ for all $i = 1, 2, 3$, where the symbol $>$ denotes the natural order “larger than” on \mathbb{R} . We summarize P in Figure 2.

	x	y	z	a	b	c
x	P			P	P	
y		P			P	P
z			P	P		P
a				P		
b					P	
c						P

Figure 2: a Partial Order P on $X \subseteq \mathbb{R}^3$.

For the poset (X, P) , $\dim(X, P) > 2$; for a proof, see for example, Sprumont (2001), Example 1 on page 438. Actually, $\dim(X, P) = 3$; one can show this either by finding three linear orders whose intersection is P or by using Hiraguchi’s inequality, $\dim(X, P) \leq |X|/2$ for $|X| \geq 4$. Given Dushnik and Miller’s theorem, P doesn’t have a conjugate. But consider another partial order Q also defined on X , where $\Delta_X \subseteq Q$ and for distinct options $u, v \in X$, uQv if and only if $u_i > v_i$ for $i = 1, 2$, and $u_i < v_i$ for $i = 3$. We present Q in the following Figure 3.

$P \cup Q$ is also a partial order. In particular, for distinct options $u, v \in X$, $u(P \cup Q)v$ if and only if $u_i > v_i$ for $i = 1, 2$. Figure 4 depicts $P \cup Q$, where we use P (instead of $P \cap Q$) to denote the diagonal.

	x	y	z	a	b	c
x		Q				
y			Q			
z		Q		Q	Q	
a					Q	Q
b						Q
c						

Figure 3: a Partial Order Q Related to P in Figure 2.

	x	y	z	a	b	c
x		P		P	P	
y			P		P	P
z		Q		P	Q	P
a					P	Q
b						P
c						

Figure 4: The Partial Order $P \cup Q$.

$P \cup Q$ has a conjugate. We use R to denote a conjugate and depict it, together with P and Q , in Figure 5 (again we use P , instead of $P \cap Q \cap R$, to denote the diagonal). R is the partial order such that for distinct options $u, v \in X$, uRv if and only if $u_1 > v_1$ and $u_2 < v_2$. Therefore for distinct u, v , $u(P \cup Q \cup R)v$ if and only if $u_1 > v_1$: $(P \cup Q) \cup R$ is a linear order.

	x	y	z	a	b	c
x		P	R		P	P
y			P		P	P
z		Q		P	Q	P
a			R		P	Q
b						P
c						

Figure 5: The Partial Order $P \cup Q$.

We found that the partial orders P and Q preserve a similar flavor to the idea “conjugates.” In particular, P and Q don’t contain any common two distinct options, that is, condition 1 (in Section 3) of conjugate is satisfied. Although under $P \cup Q$, not all distinct options are comparable, $P \cup Q$ is a partial order. That is, if condition 2 of conjugate is extended to “partial order,” P and Q will satisfy it. Finally, $P \cup Q^d$ satisfies a similar but not identical extension: $P \cup Q^d$ is not a linear order, but its transitive closure, $T_{P \cup Q^d}$, is a partial order. We generalize the idea in the following definition.

Definition 1 (Partial-conjugate). Let (X, P) and (X, Q) be two posets with the same ground set. Q is called a *partial-conjugate* of P if:

- (i) every two distinct options of X is ordered in at most one of them;
- (ii) $P \cup Q$ is a partial order;
- (iii) $T_{P \cup Q^d}$, the transitive closure of $P \cup Q^d$, is a partial order.

Remark. If Q is a partial-conjugate of P , then P is also a partial-conjugate of Q .⁵

In Definition 1, we list conditions (i), (ii), and (iii) analogous to conditions 1, 2, and 3 in Section 3. Similar to the conditions in Section 3, the three conditions here are not independent (condition (ii) and (iii) together will imply condition (i)). Condition (i) preserves condition 1 of conjugate (in Section 3) and requires empty intersection of a partial order and its partial-conjugates on comparing any two distinct options. Condition (ii) extends condition 2 of conjugate in the sense that the union of a partial order and its partial-conjugate satisfies transitivity but not necessarily completeness. Similarly, condition (iii) extends condition 3 of conjugate and requires the union of a partial order and the dual of its partial-conjugate to be transitive in the weaker sense that the transitive closure of the union is a partial order. Our next definition completes the extension of conjugate concept to use a sequence of partial orders having partial-conjugates relation so that all distinct options can be ordered under the union of the partial orders of the sequence.

Definition 2 (Sequence of Recursive Partial-conjugates). Let $(X, P_1), \dots, (X, P_n)$ be a sequence of posets with the same ground set. P_1, \dots, P_n is called a *sequence of recursive partial-conjugates* if:

- (i) for any k such that $2 \leq k \leq n - 1$, P_k is a partial-conjugate of $\cup_{i=1}^{k-1} P_i$;
- (ii) P_n is a conjugate of $\cup_{i=1}^{n-1} P_i$.

For instance, in Example 2, the sequence of three partial orders, P_1, P_2, P_3 , where $P_1 = P$, $P_2 = Q$, and $P_3 = R$, is a sequence of recursive partial-conjugates.

⁵To see this, note that $(T_{Q \cup P^d})^d = T_{(Q \cup P^d)^d} = T_{Q^d \cup P}$. Since $T_{Q^d \cup P}$ is a partial order, given that Q is a partial-conjugate of P , $T_{Q \cup P^d}$ is also a partial order.

For any poset (X, P) , if $P = P_1$ and P_1, \dots, P_n is a sequence of recursive partial-conjugates, it is possible to split a partial order of the sequence, say P_2 , into two partial orders that are partial-conjugates, and the new sequence is also a sequence of recursive partial-conjugates. Therefore, we are more interested in a sequence of recursive partial-conjugates with the smallest number of partial orders. The following definition serves this purpose.

Definition 3 (an n -fold Partial Order). Let (X, P) be a poset. The partial order P is n -fold if n is the smallest integer such that there exists a sequence of recursive partial-conjugates P_1, \dots, P_n where $P_1 = P$.

Remark 1. Let (X, P) be a poset. If P is n -fold and P_1, \dots, P_n is a sequence of recursive partial-conjugates where $P_1 = P$, then $P_1 \cup P_2$ is $(n - 1)$ -fold.

Remark 2. Let (X, P) be a poset. If P is n -fold and P_1, \dots, P_n is a sequence of recursive partial-conjugates where $P_1 = P$, then $P_k \cup P_{k+1}$ is not a partial order for any integer k such that $1 < k < n$. (Otherwise, take the union of $P_k \cup P_{k+1}$ and the number of sequence can be reduced by 1, contradiction to that P is n -fold.)

So a 2-dimensional partial order is 2-fold. The partial order in Example 2, which is 3-dimensional, is 3-fold.

Theorem 1. Let (X, P) be a poset. Then $\dim(X, P) = n$ if and only if P is n -fold, i.e.,

- A. If $\dim(X, P) = n$, then P is at most n -fold;
- B. If P is n -fold, then $\dim(X, P) \leq n$.

4.1 Proof of Theorem 1A

We show: If $\dim(X, P) = n$, then P is at most n -fold.

Proof. Consider a poset (X, P) and suppose that $\dim(X, P) = n$. Since $\dim(X, P) = n$, there exist n linear orders L_1, \dots, L_n such that

$$P_1 = P = L_1 \cap \dots \cap L_n.$$

In what follows, we will only use P_1 to denote both P and P_1 .

We show that P_1 is at most n -fold by constructing a sequence of recursive partial-conjugates P_1, \dots, P_n .

Define:

$$P_2 := L_1 \cap \dots \cap (L_n)^d$$

$$P_3 := L_1 \cap \dots \cap (L_{n-1})^d$$

$$\begin{aligned} & \vdots \\ P_n &:= L_1 \cap (L_2)^d. \end{aligned}$$

We show that (i) for any k such that $2 \leq k \leq n-1$, P_k is a partial-conjugate of $\bigcup_{i=1}^{k-1} P_i$; (ii) P_n is a conjugate of $\bigcup_{i=1}^{n-1} P_i$, and therefore, P_1, \dots, P_n is a sequence of recursive partial-conjugates. For any k such that $2 \leq k \leq n-1$, since

$$\begin{aligned} \bigcup_{i=1}^{k-1} P_i &= P_1 \cup P_2 \cup \dots \cup P_{k-1} \\ &= (L_1 \cap \dots \cap L_n) \cup \left(L_1 \cap \dots \cap (L_n)^d \right) \\ &\quad \cup \dots \cup \left(L_1 \cap \dots \cap (L_{n-k+3})^d \right) \\ &= L_1 \cap \dots \cap L_{n-k+2} \end{aligned}$$

and

$$P_k = L_1 \cap \dots \cap (L_{n-k+2})^d,$$

every pair of distinct options of X is ordered in at most one of them and $(\bigcup_{i=1}^{k-1} P_i) \cup P_k = L_1 \cap \dots \cap L_{n-k+1}$, which is a partial order. Additionally, since $\bigcup_{i=1}^{k-1} P_i = L_1 \cap \dots \cap L_{n-k+2} \subseteq L_{n-k+2}$, and $P_k = L_1 \cap \dots \cap (L_{n-k+2})^d$, which implies $(P_k)^d \subseteq L_{n-k+2}$, $(\bigcup_{i=1}^{k-1} P_i) \cup (P_k)^d \subseteq L_{n-k+2}$. Therefore, $T_{(\bigcup_{i=1}^{k-1} P_i) \cup (P_k)^d}$, the transitive closure of $(\bigcup_{i=1}^{k-1} P_i) \cup (P_k)^d$, is a partial order. So, P_k is a partial-conjugate of $\bigcup_{i=1}^{k-1} P_i$. It is also obvious that P_n is a conjugate of $\bigcup_{i=1}^{n-1} P_i$ since $\bigcup_{i=1}^{n-1} P_i = L_1 \cap L_2$ and $P_n = L_1 \cap (L_2)^{-1}$.

So we have constructed a sequence of recursive partial-conjugates P_1, \dots, P_n where $P_1 = P$. And therefore, P is at most n -fold. \square

4.2 Proof of Theorem 1B

We show: If P is n -fold, then $\dim(X, P) \leq n$.

Proof. Since P is n -fold, consider a sequence of recursive partial-conjugates P_1, \dots, P_n where $P_1 = P$. We first show that for any $2 \leq k \leq n-1$, if there exist m linear orders such that

$$\bigcup_{i=1}^k P_i = L_1 \cap L_2 \cap \dots \cap L_m$$

then we can find another linear order, denoted as L_{m+1} , such that

$$\bigcup_{i=1}^{k-1} P_i = L_1 \cap L_2 \cap \dots \cap L_m \cap L_{m+1}.$$

To see this, suppose $\bigcup_{i=1}^k P_i = L_1 \cap L_2 \cap \dots \cap L_m$ for linear orders L_1, \dots, L_m . Since P_1, \dots, P_n is a sequence of recursive partial-conjugates, P_k is a partial-conjugate of $\bigcup_{i=1}^{k-1} P_i$. By condition (iii) of Definition 1, $T_{(\bigcup_{i=1}^{k-1} P_i) \cup (P_k)^d}$, the

transitive closure of $(\cup_{i=1}^{k-1} P_i) \cup (P_k)^d$, is a partial order. Therefore, it can be extended to a linear order, denoted as L_{m+1} . Since

$$\cup_{i=1}^k P_i = L_1 \cap L_2 \cap \cdots \cap L_m$$

and

$$(\cup_{i=1}^{k-1} P_i) \cup (P_k)^d \subseteq L_{m+1}$$

we have

$$\cup_{i=1}^{k-1} P_i = L_1 \cap L_2 \cap \cdots \cap L_m \cap L_{m+1}$$

given that $(P_k)^d$ is the dual of P_k . So we have found another linear order L_{m+1} such that $\cup_{i=1}^{k-1} P_i = L_1 \cap L_2 \cap \cdots \cap L_m \cap L_{m+1}$. Since P_1, \dots, P_n is a sequence of recursive partial-conjugates, P_n is a conjugate of $\cup_{i=1}^{n-1} P_i$. Therefore, $\cup_{i=1}^{n-1} P_i$ is at most dimension 2 and there exist two linear orders L_1 and L_2 such that

$$\cup_{i=1}^{n-1} P_i = L_1 \cap L_2.$$

Give the result we have just proved, there exists a third linear order L_3 , such that

$$\cup_{i=1}^{n-2} P_i = L_1 \cap L_2 \cap L_3.$$

Repeating the same process, there exists a number of linear orders L_4, \dots, L_n such that

$$\cup_{i=1}^{n-3} P_i = L_1 \cap L_2 \cap L_3 \cap L_4$$

$$\vdots$$

$$P_1 = P = L_1 \cap L_2 \cap \cdots \cap L_n$$

so, $\dim(X, P) \leq n$. \square

5 Discussion

Extending the work by Dushnik and Miller, we introduce some concepts related to their conjugate idea and provide a characterization of a partial order having dimension n in general. However, as in Dushnik and Miller (1941) and pointed out by Sprumont (2001), our characterization result is of an “existential” nature so that finding the objects (a partial-conjugate and a sequence of recursive partial-conjugates here) stated in our characterization is not necessarily easier than finding the dimension of the partial order. Since the characterization of an n -dimensional partial order for any given number of n has been open, the current work hopes to shed some light on that question. A characterization that consists of some explicit and simpler conditions which can be easier to test and applied remains an interesting, though challenging, problem.

References

- [1] K. Baker, P. Fishburn, F. Roberts, Partial orders of dimension 2, *Networks* 2 (1972) 11-28.
- [2] T. Demuyne, The computational complexity of rationalizing Pareto optimal choice behavior, *Social Choice and Welfare*, forthcoming.
- [3] B. Dushnik, E. Miller, Partially ordered sets, *American Journal of Mathematics* 63 (1941) 600-610.
- [4] F. Echenique, L. Ivanov, Implications of pareto efficiency for two-agent (household) choice, *Journal of Mathematical Economics* 47 (2011) 129-136.
- [5] T. Hiraguchi, On the dimension of partially ordered sets, *Science Reports of Kanazawa University* 1 (1951) 77-94.
- [6] D. Kelly, The 3-irreducible partially ordered sets, *Canadian Journal of Mathematics* XXIX (1977) 367-383.
- [7] S. Qi, Paretian partial orders: the two-agent case, *Mimeo*.
- [8] Y. Sprumont, Paretian quasi-orders: the regular two-agent case, *Journal of Economic Theory* 101 (2001) 437-456.
- [9] W. Trotter, *Combinatorics and partially ordered sets*, Johns Hopkins University Press, 1992.
- [10] W. Trotter, J. Moore, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, *Discrete Mathematics* 16 (1976) 361-381.
- [11] M. Yannakakis, The complexity of the partial order dimension problem, *SIAM J. Alg. Disc. Meth.* 3 (1982) 351-358.